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THREE SPACE PROPERTIES AND BASIS EXTENSIONS

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ABSTRACT

We discuss and prove three space properties and basis extension theorems of the following kind:

Let Y be a separable L_1-space and $X \subset Y$ a non-reflexive subspace such that Y/X has a basis. Then X has a basis.

If Y is a separable $C(K)$-space and $X \subset Y$ is such that Y/X is non-reflexive, then every basis of X can be extended to a basis of Y.

1. Introduction

Let Y be a separable (real or complex) Banach space and $X \subset Y$ a closed subspace. We discuss the following question:

If two of the three Banach spaces Y, X and Y/X have bases, does it follow that the third one has a basis, too?

The answer to all variants of this question is *no* in general. For example, there exists a separable Banach space Y without basis which contains an isomorphic copy of c_0, [2]. It was shown in [6] that there is a subspace $X \subset Y$ with basis where Y/X has a basis, too.

Moreover, there is a separable Banach space W without basis which is complemented in a space with basis (i.e. has the bounded approximation property), [10], [8]. Then, by [6], there is a Banach space V with basis such that $Y = W \oplus V$ has a basis. So, if we put $X = V$ then Y and X have bases but Y/X fails to have

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a basis. Finally, if $X = W$, then Y and Y/X have bases but X does not possess a basis.

Closely related to three space problems is the following basis extension problem: Let X, Y be Banach spaces with bases, such that $X \subset Y$, and assume that Ω_X is a basis of X. Is there a basis Ω of Y containing Ω_X as a subsequence?

The answer again is *no* in general since a positive answer would imply that Y/X always has a basis, too.

The aim of this paper is to give some positive answers to the preceding questions if Y is a $\mathcal{L}_\infty$- or a $\mathcal{L}_1$-space.

For two isomorphic Banach spaces U, V let $d(U,V)$ be the Banach–Mazur distance, i.e.

$$d(U,V) = \inf\{\|T\| \cdot \|T^{-1}\| \colon T\colon U \to V \text{ is an (onto-) isomorphism}\}.$$

Let $1 \le p \le \infty$. Y is called a $\mathcal{L}_p$-space if there is $\lambda > 0$ such that, for each finite dimensional subspace $E \subset Y$, there exists another finite dimensional subspace $F \subset Y$ with $E \subset F$ and $d(F, l_p^{\dim F}) < \lambda$.

It is well-known that separable $\mathcal{L}_p$-spaces have bases, [3].

THEOREM:

(a) *Let Y be a separable $\mathcal{L}_\infty$-space and $X \subset Y$ a closed subspace. Assume that $\{x_j\}_{j=1}^\infty$ is a basis of X and $\{y_j\}_{j=1}^\infty$ is a basis of Y. Then $Y \oplus c_0$ has a basis $\{z_k\}_{k=1}^\infty$ containing $\{x_j\}_{j=1}^\infty$ as a subsequence. Furthermore, $\{z_k\}_{k=1}^\infty$ has another subsequence which is equivalent to $\{y_j\}_{j=1}^\infty$. (Here Y is identified with the left-hand summand of $Y \oplus c_0$.)*

(b) *Let Y be a separable $\mathcal{L}_1$-space and assume that $\{w_j\}_{j=1}^\infty$ is a basis of Y/X for some subspace $X \subset Y$. Then there is a basis $\{z_k\}_{k=1}^\infty$ of $Y \oplus l_1$ and a subsequence $\Lambda = \{k_j\}_{j=1}^\infty$ of the indices such that*

$$qz_k = \begin{cases} w_j & \text{if } k = k_j, \\ 0 & \text{if } k \notin \Lambda. \end{cases}$$

Here $q\colon Y \oplus l_1 \to Y/X$ is the map with $q(y + l) = y + X$, $y \in Y$, $l \in l_1$.

Note that the basic sequence $\{x_j\}_{j=1}^\infty$ in Theorem (a) is a subsequence of $\{z_k\}_{k=1}^\infty$, not just equivalent to a subsequence.

We prove the theorem in Sections 2 and 3. The proof of the theorem shows that the basis constant for the basis in $Y \oplus c_0$ (or $Y \oplus l_1$) depends only on the basis constant of the given basis in X (or Y/X, resp.) and the basis constant of the space Y.

Here we discuss some corollaries and examples. As an immediate consequence of the theorem we obtain

COROLLARY 1:

(a) *Let Y be a separable $\mathcal{L}_\infty$-space and assume that the subspace $X \subset Y$ has a basis. Then $(Y/X) \oplus c_0$ has a basis.*

(b) *Let Y be a separable $\mathcal{L}_1$-space and assume that Y/X has a basis for some subspace $X \subset Y$. Then $X \oplus l_1$ has a basis.*

Let Y be a separable $C(K)$-space and $X \subset Y$ such that Y/X is non-reflexive. Assume that $q\colon Y \to Y/X$ is the quotient map. It is well known, [1], that there is a subspace $U \subset Y$, $U \sim c_0$, where $q|_U$ is an isomorphism. So, U is complemented in Y and qU is complemented in Y/X.

Furthermore, if Y is a separable L_1-space, then every non-reflexive subspace $X \subset Y$ contains a complemented copy of l_1, [4].

Hence we obtain a slight extension of the theorem for $C(K)$- and L_1-spaces:

COROLLARY 2:

(a) *Let Y be a separable $C(K)$-space and let $\{y_j\}_{j=1}^\infty$ be a basis of Y. Assume $X \subset Y$ is such that Y/X is non-reflexive. Then every basis of X can be extended to a basis of Y containing another subsequence which is equivalent to $\{y_j\}_{j=1}^\infty$.*

(b) *Let Y be a separable L_1-space and let $X \subset Y$ be a non-reflexive subspace. If Ω_0 is a basis of Y/X then there is a basis Ω of Y such that $q(\Omega) = \Omega_0$ where $q\colon Y \to Y/X$ is the quotient map.*

COROLLARY 3:

(a) *Let Y be a separable $C(K)$-space and $X \subset Y$ a subspace with basis such that Y/X is non-reflexive. Then Y/X has a basis.*

(b) *Let Y be a separable L_1-space and $X \subset Y$ a non-reflexive subspace such that Y/X has a basis. Then X has a basis.*

Examples: (1) Any closed subspace $X \subset l_1$ is non-reflexive or finite-dimensional. So, if l_1/X has a basis, then X must have a basis. Recall that every separable Banach space is a quotient of l_1.

(2) Let $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$. Recall that every separable Banach space can be isomorphically embedded into $C(\mathbb{T})$. If $X \subset C(\mathbb{T})$ is such that $C(\mathbb{T})/X$ is non-reflexive and X has a basis, then any basis of X can be extended to a basis Ω of $C(\mathbb{T})$. It is known that $C(\mathbb{T})$ has a basis Ω_0 such that every basic sequence of any separable Banach space is equivalent to a subsequence of Ω_0 ([7]). So, using

Corollary 2 we can even arrange Ω such that in addition any basic sequence is equivalent to a subsequence of Ω. This is true, in particular, if $X = A$ is the disc-algebra.

(3) Let $\Lambda \subset \mathbb{Z}$ and let

$$C_\Lambda = \text{ closed span of } \{z^n : n \in \Lambda\} \subset C(\mathbb{T}).$$

It is unknown if C_Λ has a basis for every $\Lambda \subset \mathbb{Z}$.

However, if C_Λ has a basis then so does $C(\mathbb{T})/C_\Lambda$ provided that $C(\mathbb{T})/C_\Lambda$ is non-reflexive.

2. The main construction

At first we present the main construction and then we prove part (a) of the theorem. Let Y be a separable $\mathcal{L}_\infty$-space, $X \subset Y$ a closed subspace, $\{x_j\}_{j=1}^\infty$ a basis of X with basis projections $T_j: X \to X$. Put $T_0 = 0$. Finally, let $R_n: Y \to Y$ be the sequence of basis projections for a given basis of Y. By counting the T_j several times if necessary (i.e. $T_1 = T_2 = \cdots = T_{j_1}$, $T_{j_1+1} = \cdots = T_{j_2}$, etc.) we may assume w.l.o.g.

$$\|(\mathrm{id} - R_{n-1})(T_n - T_{n-1})\| \leq 2^{-n} \quad \text{for all } n. \tag{2.1}$$

Put $R_0 = 0$. Since Y is a separable $\mathcal{L}_\infty$-space we find subspaces $E_1 \subset E_2 \subset \cdots$ satisfying

$$R_n Y \cup T_n X \subset E_n, \quad \overline{\bigcup E_n} = Y \quad \text{and} \quad \sup_n d(E_n, l_\infty^{\dim E_n}) < \infty. \tag{2.2}$$

Let Z be the completion of the space of all finite sequences $\{e_k\}_{k=1}^\infty$, where $e_k \in E_k$, under the norm

$$\|\{e_k\}_{k=1}^\infty\| = \sup_n \|\sum_{k=1}^n e_k\|. \tag{2.3}$$

Identify $y \in Y$ with $\{(R_n - R_{n-1})y\}_{n=1}^\infty$ in Z. Let

$$P\{e_k\}_{k=1}^\infty = \{(R_n - R_{n-1})\sum_k e_k\}_{n=1}^\infty. \tag{2.4}$$

Then $P: Z \to Y$ is a bounded projection. Finally, put

$$\tilde{X} = \{\{(T_k - T_{k-1})x\}_{k=1}^\infty \in Z : x \in X\}. \tag{2.5}$$

Then $\tilde{X}$ is isomorphic to X.

2.1. LEMMA: *$\ker P$ is spanned by the elements*

$$e(n) := (\underbrace{0,\dots,0}_{n-1}, e, -e, 0, \dots), \quad e \in E_n, \quad n = 1, 2, \dots. \tag{2.6}$$

So, $\ker P = (\sum \oplus E_n)_{(0)} \sim c_0$, $Z \sim Y \oplus c_0$ and P corresponds to the canonical projection $Y \oplus c_0 \to Y$.

Proof: (2.3) and (2.6) yield, for $e_k \in E_k$, $\|\sum_k e_k(k)\| = \sup_n \|e_n\|$. Clearly, $e_k(k) \in \ker P$ for all k. Moreover, the elements $e_k(k), k = 1, 2, \dots$, span $(\mathrm{id} - P)Z$. Indeed, consider $e \in E_n$ and

$$z = (\underbrace{0,\dots,0}_{n-1}, e, 0, \dots).$$

Put, for $m = 1, 2, \dots,$

$$z_m = -\sum_{k=1}^{n+m-1} (R_k e)(k) + (\underbrace{0,\dots,0}_{n-1}, e, \underbrace{0,\dots,0}_{m-1}, -e, 0, \dots).$$

Then, by (2.4),

$$(\mathrm{id} - P)z - z_m =$$
$$(\underbrace{0,\dots,0}_{n+m-1}, e - R_{n+m}e, -(R_{n+m+1} - R_{n+m})e, -(R_{n+m+2} - R_{n+m+1})e, \dots).$$

Since $\lim_{k\to\infty} R_k e = e$ we conclude, using (2.3), $\lim_{m\to\infty} \|(\mathrm{id} - P)z - z_m\| = 0$. This implies that $\ker P$ is the closed linear span of the elements $e_k(k)$, $e_k \in E_k$, $k = 1, 2, \dots$. Hence

$$Z = PZ \oplus \ker P \sim Y \oplus (\sum \oplus E_k)_{(0)} \sim Y \oplus c_0. \qquad \blacksquare$$

2.2. LEMMA: *There is a bounded projection $\tilde{P}$ from Z onto a subspace $\tilde{Y} \subset Z$ with*

$$\ker P = \ker \tilde{P} \quad \text{and} \quad \tilde{P}X = \tilde{X}. \tag{2.7}$$

Proof: Let $\rho: Z \to Z/\tilde{X}$ be the quotient map and put

$$Q(\{e_k\}_{k=1}^{\infty} + \tilde{X}) = \sum_k e_k + X.$$

Then, in view of (2.5), Q is a well defined contractive map from $Z/\tilde{X}$ to Y/X. (It is easily seen that Q is even a quotient map.)

We determine $\ker Q$. To this end let $Q(\{e_k\}_{k=1}^{\infty} + \tilde{X}) = X$, i.e.

$$x = \sum_k e_k \in X. \tag{2.8}$$

Put $f_n = e_1 + \cdots + e_n - T_n x$. Then $f_n \in E_n$ and, by (2.6),

$$\sum_{k=1}^{n} f_k(k) =$$

$$(e_1 - (T_1 - T_0)x, e_2 - (T_2 - T_1)x, \ldots, e_n - (T_n - T_{n-1})x, -f_n, 0, \ldots).$$

Hence

$$\|\{e_k\}_{k=1}^{\infty} - \sum_{k=1}^{n} f_k(k) + \tilde{X}\| \le \sup_m \|\sum_{k=1}^{n+m} e_k - T_{n+m}x\|,$$

i.e., in view of (2.8), $\lim_n \|\{e_k\}_{k=1}^{\infty} - \sum_{k=1}^{n} f_k(k) + \tilde{X}\| = 0$. Clearly, $f_k(k) + \tilde{X} \in \ker Q$ for all k. This proves that $\ker Q$ is spanned by the elements of the form $g_n(n) + \tilde{X}$, $g_n \in E_n$. We obtain $\|\sum_{k=1}^{n} g_k(k) + \tilde{X}\| \le \sup_k \|g_k\|$ and

$$\begin{aligned}\|\sum_{k=1}^{n} g_k(k) + \tilde{X}\| &\ge \frac{1}{2} \inf_x (\sup_{k\le n} \|g_k + T_k x\| + \sup_m \|T_{n+m}x\|) \\ &\ge \frac{1}{2c} \sup_{k\le n} \|g_k\|\end{aligned}$$

since $T_k T_{n+m} = T_k$ if $k \le n$. Here $c = \sup_m \|T_m\|$. This shows that $\ker Q \sim c_0$. Moreover, using (2.3) and (2.6) we see that $\rho|_{\ker P}$ is an isomorphism onto $\ker Q$. Since $\ker Q \sim c_0$ we find a bounded projection S: $Z/\tilde{X} \to \ker Q$. Put

$$\tilde{P} = \mathrm{id} - (\rho|_{\ker P})^{-1} S\rho.$$

Then $\tilde{P}$: $Z \to Z$ is a bounded projection and we obtain

$$\ker \tilde{P} = (\rho|_{\ker P})^{-1} S\rho Z = \ker P.$$

Finally we claim $\tilde{P}X = \tilde{X}$. At first we observe $\tilde{P}|_{\tilde{X}} = \mathrm{id}$. If $x \in X$ then $Q\rho\{(R_n - R_{n-1})x\}_{k=1}^{\infty} = 0$. Hence we have

$$\begin{aligned}S\rho\{(R_n - R_{n-1})x\}_{n=1}^{\infty} &= \rho\{(R_n - R_{n-1})x\}_{n=1}^{\infty} \\ &= \rho(\{(R_n - R_{n-1})x\}_{n=1}^{\infty} - \{(T_n - T_{n-1})x\}_{n=1}^{\infty}).\end{aligned}$$

(2.4) implies

$$w := \{(R_n - R_{n-1})x\}_{n=1}^{\infty} - \{(T_n - T_{n-1})x\}_{n=1}^{\infty} \in \ker P.$$

Thus

$$\tilde{P}\{(R_n - R_{n-1})x\}_{n=1}^{\infty} = \{(R_n - R_{n-1})x\}_{n=1}^{\infty} - w = \{(T_n - T_{n-1})x\}_{n=1}^{\infty}.$$

This implies $\tilde{P}X = \tilde{X}$. ∎

2.3. Corollary: *Put $\tilde{Y} = \tilde{P}Y$. Then $\tilde{P}|_Y$ is an isomorphism between Y and $\tilde{Y}$ where $(\tilde{P}|_Y)^{-1} = P|_{\tilde{Y}}$. In particular $Z \sim \tilde{Y} \oplus c_0$ and $\tilde{X} \subset \tilde{Y}$ corresponds to the given embedding $X \subset Y$.*

Let $m_n = \sum_{k=1}^{n} \dim E_k$ and put, if $e_k \in E_k$,

$$P_{m_n}\{e_k\}_{k=1}^{\infty} = (e_1, \ldots, e_n, 0, \ldots). \tag{2.9}$$

Clearly, $\{P_{m_n}\}_{n=1}^{\infty}$ is an FDD-sequence, i.e. $P_{m_k}P_{m_j} = P_{m_{\min(j,k)}}$ for all j,k and $\lim_{n\to\infty} P_{m_n}z = z$ for all $z \in Z$.

Let $\{y_j\}_{j=1}^{\infty}$ be the basis of Y corresponding to the given basis projections R_j. Regard y_j as elements of Z. Recall, $\{x_j\}_{j=1}^{\infty}$ is the basis of X corresponding to the projections T_k, i.e. $T_{j_k}\sum_{i=1}^{\infty}\alpha_i x_i = \sum_{i=1}^{k}\alpha_i x_i$ for suitable indices j_k. Remember that we assumed $T_{j_k} = T_{j_k+1} = \cdots = T_{j_{k+1}-1}$.

2.4. Lemma: *Let $\tilde{x}_j \in \tilde{X}$ correspond to $x_j \in X$. Then there is a basis $\{z_k\}_{k=1}^{\infty}$ of Z which contains $\{\tilde{x}_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ as subsequences. Moreover, if P_k are the basis projections of $\{z_k\}_{k=1}^{\infty}$ then P_{m_n} coincides with the projection in* (2.9).

Proof: We retain the notation of j_k preceding Lemma 2.4. By (2.5) we obtain

$$\tilde{x}_k = (\underbrace{0, \ldots, 0}_{j_k - 1}, x_k, 0, \ldots).$$

In view of (2.2) we find a basis $\{e_{n,j}\}_{j=1}^{k_n}$ of E_n whose basis constant does not depend on n such that $e_{n,1} = y_n$ and, in the case that $n = j_k$, $e_{n,2} = x_k$. Note that, in view of (2.1), (2.2), we can write

$$E_n = \begin{cases} \tilde{E}_n \oplus \operatorname{span}\{y_n\} \oplus \operatorname{span}\{x_k\}, & \text{if } n = j_k \text{ for some } k \\ \tilde{E}_n \oplus \operatorname{span}\{y_n\}, & \text{else} \end{cases}$$

for some suitable one- or two-codimensional subspace $\tilde{E}_n$ of E_n.

We have $m_{n-1} = \sum_{j=1}^{n-1} k_j$. Put

$$z_{m_{n-1}+j} = (\underbrace{0, \dots, 0}_{n-1}, e_{n,j}, 0, \dots).$$

According to (2.3) we obtain a basis of Z which contains $\{\tilde{x}_j\}_{j=1}^\infty$ and $\{y_k\}_{k=1}^\infty$ as subsequences. The basis projections clearly satisfy (2.9). ∎

Corollary 2.3 and Lemma 2.4 prove part (a) of the theorem. The basis constant of $\{z_k\}_{k=1}^\infty$ depends only on $\sup_n \|R_n\|$ and $\sup_n \|T_n\|$. Virtually the same construction can be used to show that the theorem and the corollaries remain true if we replace 'basis' by 'FDD' or other bounded approximation properties.

We need another lemma to prove part (b) of the theorem. Let z_k^* be the biorthogonal functionals of the z_k. We retain the notation of the basis $\{x_j\}_{j=1}^\infty$ of X. Denote the corresponding basis in $\tilde{X}$ by $\{\tilde{x}_j\}_{j=1}^\infty$ (i.e. $\tilde{x}_j = \tilde{P}x_j$) and let $\tilde{x}_j^*$ be the biorthogonal functionals of the $\tilde{x}_j$. Again, let m_n be the indices of (2.9).

2.5. Lemma: *Let P_m be the basis projections of the basis $\{z_m\}_{m=1}^\infty$ of Lemma 2.4. Put $U =$ norm closure of $(\bigcup_{m=1}^\infty P_m^* Z^*)$. Then $U \sim l_1$. Moreover, let $v_j \in Z^*$ be given elements such that $v_j|_{\tilde{X}} \in \operatorname{span}\{\tilde{x}_k^*: k = 1, 2, \dots\}$ for all j. Then there is a subsequence $\{n_k\}_{k=1}^\infty$ of the indices satisfying*

$$v_j|_{\tilde{X}} = (P_{m_{n_j}}^* v_j)|_{\tilde{X}}, \quad j = 1, 2, \dots. \tag{2.10}$$

Proof: For $e^* \in E_n^*$ put

$$(e^*)_n(\{e_k\}_{k=1}^\infty) = e^*(\sum_{k=1}^n e_k).$$

Let $U =$ normed closed span of $\{(e^*)_n: e^* \in E_n^*, n = 1, 2, \dots\}$. We have, by (2.9),

$$P_{m_n}^*(e^*)_k = \begin{cases} (e^*)_k & \text{if } k \le n, \\ (e^*|_{E_n})_n & \text{if } k > n. \end{cases}$$

Hence $P_{m_n}^* U \subset U$. Moreover, by definition of P_{m_n}, $P_{m_n}^* Z^* \subset U$. So, norm closure of $\bigcup_{m=1}^\infty P_m^* Z^* = U$. For $e_m^* \in E_m^*$ we obtain, by (2.6),

$$\begin{aligned} \|\sum_m (e_m^*)_m|_{\ker P}\| &= \sup_{\|\sum e_k(k)\| \le 1} |\sum_m e_m^*(e_m)| \\ &= \sup\{|\sum_m e_m^*(e_m)|: e_m \in E_m, \|e_m\| \le 1 \text{ for all } m\} \\ &= \sum_m \|e_m^*\|. \end{aligned}$$

This proves that the restriction map r: $Z^* \to (\ker P)^*$ maps U isomorphically onto $(\ker P)^* \sim l_1$. Hence $U \sim l_1$.

Now, let $v_j \in Z^*$ be such that $v_j|_{\tilde{X}} \in \operatorname{span}\{\tilde{x}_k^*: k = 1, 2, \ldots\}$. This means by (2.5) that

$$v_j(\tilde{x}_k) = 0 \quad \text{if } k \geq n_j \text{ for suitable } n_j.$$

Since the P_m are the basis projections for $\{z_m\}_{m=1}^\infty$ and this basis contains $\{\tilde{x}_k\}_{k=1}^\infty$ as a subsequence, we obtain

$$v_j(\tilde{x}_k) = (P_{m_{n_j}}^* v_j)(\tilde{x}_k) \quad \text{for all } k.$$

This implies (2.10). ∎

3. Proof of Theorem (b)

We want to retain the notation of Section 2, so we change the notation of Theorem (b). Let V be a separable $\mathcal{L}_1$-space. Then V always has a basis ([3]), say $\{v_j\}_{j=1}^\infty$. Let $W \subset V$ be a closed subspace such that V/W has a basis $\{w_j\}_{j=1}^\infty$. We can arrange the v_j such that

$$\text{(3.1)} \qquad v_j + W \in \operatorname{span}\{w_k: k = 1, 2, \ldots\} \quad \text{for all } j.$$

Denote by v_j^* and w_j^* the corresponding biorthogonal functionals. Put

$$\text{(3.2)} \qquad X = \text{ normed closed span of } \{w_k^*\}_{k=1}^\infty.$$

Then

$$X \subset (V/W)^* = W^\perp \subset V^*.$$

Let T_n be the basis projections for the basis $\{w_k^*\}_{k=1}^\infty$ of X. Put $x_k = w_k^*$ and obtain $x_k^* = w_k$. We want to apply the main construction of Section 2 to X and the T_k. Therefore we need Y. At first, observe that V^* is a $\mathcal{L}_\infty$-space ([5]). Let $Y \subset V^*$ be a separable $\mathcal{L}_\infty$-space containing X and $\{v_k^*\}_{k=1}^\infty$.

Let Z, U, z_k^*, P_k, $\tilde{X}$, $\tilde{x}_k^*$, P, $\tilde{P}$ be defined as in Section 2 and let r: $Z^* \to \tilde{X}^*$ be the restriction map. Lemmas 2.4 and 2.5 imply $U \sim l_1$ and

$$\text{(3.3)} \qquad r(z_k^*) = \begin{cases} \tilde{x}_j^* & \text{if } k = k_j \\ 0 & \text{if } k \notin \Lambda \end{cases}$$

for a suitable subsequence $\Lambda = \{k_j\}_{j=1}^\infty$. Moreover, $\{z_k^*\}_{k=1}^\infty$ is a basis of U. (3.2) and (3.3) yield $r(U) \sim V/W$. Every $y \in Y \subset V^*$ is a functional on V. For $z \in Z$ we have $Pz \in Y$ and $Pz = P\tilde{P}z$ (Corollary 2.3). Put

$$\text{(3.4)} \qquad \tilde{v}_j(z) = (Pz)(v_j), \quad j = 1, 2, \ldots.$$

Then $\tilde{v}_j \in Z^*$ and $\tilde{v}_j(\tilde{P}z) = (Pz)(v_j)$ for all $z \in Z$. Clearly,

$$\tilde{V} := \text{ normed closed span of } \{\tilde{v}_j : j = 1, 2, \ldots\} \sim V.$$

In view of (3.1) and (3.4) we obtain that

$$\tilde{v}_j|_{\tilde{X}} \in \operatorname{span}\{\tilde{x}_k^* : k = 1, 2, \ldots\} \quad \text{for all } j.$$

Lemma 2.5 applied to $\tilde{v}_j$ instead of v_j yields indices n_j satisfying (2.10). Put $u_j = P^*_{m_{n_j}} \tilde{v}_j$. Then we consider $\tilde{V} \oplus U \sim V \oplus U$. Using (2.10) we conclude that

$$(0, z_1^*), \ldots, (0, z^*_{m_{n_1}}), (\tilde{v}_1, -u_1), (0, z^*_{m_{n_1}+1}), \ldots, (0, z^*_{m_{n_2}}), (\tilde{v}_2, -u_2), (0, z^*_{m_{n_2}+1}), \ldots$$

is a basis of $\tilde{V} \oplus U$. This follows from the fact that $\{\tilde{v}_j\}_{j=1}^\infty$ is a basic sequence in Z^* and the $P^*_{m_{n_j}}$ are basis projections for the basic sequence $\{z_k^*\}_{k=1}^\infty$. In particular we have, for all α_k,

$$\sum_{k \geq j} \alpha_k P^*_{m_{n_j}} u_k = \sum_{k \geq j} \alpha_k P^*_{m_{n_j}} \tilde{v}_k.$$

If we put $r(\tilde{v}, u) = r\tilde{v} + ru$, $\tilde{v} \in \tilde{V}$, $u \in U$, then we have

$$r(\tilde{v}_j, -u_j) = 0. \tag{3.5}$$

This is a consequence of (2.10). Finally, since $U \sim l_1$, we find a bounded linear operator $T\colon U \to \tilde{V}$ with $rTu = u|_{\tilde{X}}$ for all $u \in U$. Put

$$\tilde{U} = \{(-Tu, u)\colon u \in U\}.$$

Then $\tilde{V} \oplus \tilde{U} = \tilde{V} \oplus U \sim V \oplus l_1$. We have $r(-Tu, u) = 0$ for all $u \in U$. This together with (3.3) and (3.5) proves Theorem (b). ∎

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